Phillips' Lemma for L-embedded Banach spaces

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Abstract. In this note the following version of Phillips' lemma is proved. The L-projection of an L-embedded space - that is of a Banach space which is complemented in its bidual such that the norm between the two complementary subspaces is additive - is weak*-weakly sequentially continuous.

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Phillips' classical lemma [9] refers to a sequence (μ_n) in ba($\mathbb{N}$) (the Banach space of finitely bounded measures on the subsets of $\mathbb{N}$) and states that if $\mu_n(A) \to 0$ for all $A \subset \mathbb{N}$ then $\sum_k |\mu_n(\{i\})| \to 0$. It is routine to interpret this result as the weak*-weak-sequential continuity of the canonical projection from the second dual of l^1 onto l^1 because this continuity together with l^1's Schur property gives exactly Phillips' lemma. (Cf., for example, [2, Ch. VII].) Therefore the following theorem generalizes Phillips' lemma (for the definitions see below):

Theorem 0.1. *The L-projection of an L-embedded Banach space is weak*-weakly sequentially continuous.*

The theorem will be proved at the end of the paper.

The theorem has been known in the two particular cases when the L-embedded space in question is the predual of a von Neumann algebra or the dual of an M-embedded Banach spece Y. In the first case the result follows from [1, Th. III.1]; in the second case Y has Pełczyński's property (V) ([3] or [4, Th. III.3.4]) and has therefore, by [4, Prop. III.3.6], what in [6, p. 73] or in [10] is called the weak Phillips property whence the result by [4, Prop. III.2.4].

Preliminaries. By definition a Banach space X is *L-embedded* (or an *L-summand in its bidual*) if there is a linear projection P on its bidual X^{**} with range X such that $\|Px^{**}\| + \|x^{**} - Px^{**}\| = \|x^{**}\|$ for all $x^{**} \in X^{**}$. The projection P is called L-projection. Throughout this note X denotes an L-embedded Banach space with L-projection P. We have the decomposition $X^{**} = X \oplus_1 X_{\rm s}$ where $X_{\rm s}$ denotes the kernel of P that is the range of the projection $Q = \mathrm{id}_{X^{**}} - P$. We recall that a series $\sum z_j$ in a Banach space Z is called *weakly unconditionally Cauchy*

(wuC for short) if $\sum |z^*(z_j)|$ converges for each $z^* \in Z^*$ or, equivalently, if there is a number M such that $\| \sum_{j=1}^n \alpha_j z_j \| \leq M \max_{1\leq j\leq n} |\alpha_j|$ for all $n \in \mathbb{N}$ and all scalars α_j. The presence of a non-trivial wuC-series in a dual Banach space is equivalent to the presence of an isomorphic copy of l^∞. For general Banach space theory and undefined notation we refer to [5], [7], or [2]. The standard reference for L-embedded spaces is [4]; here we mention only that besides the Hardy space H^1 the preduals of von Neumann algebras - hence in particular $L^1(\mu)$-spaces and l^1 - are L-embedded. Note in passing that in general an L-embedded Banach space, contrary to l^1, need not be a dual Banach space.

The proof of the theorem consists of two halves. The first one states that the L-projection sends a weak*-convergent sequence to a relatively weakly sequentially compact set. This has already been proved in [8]. The second half asserts the existence of the 'right' limit and can be deduced from the corollary below which states that the singular part X_{s} of the bidual is weak*-sequentially closed. Note that X_{s} is weak*-closed if and only if X is the dual of an M-embedded Banach space [4, IV.1.9]. The following lemma contains the two main ingredients for the proof of the theorem namely two wuC-series $\sum x_k^*$ and $\sum u_k^*$ by means of which the theorem above will reduce to Phillips' original lemma. The first one has already been constructed in [8], the construction of the second one is (somewhat annoyingly) completely analogous, with the rôles of P and Q interchanged, cf. (0.20) and (0.21). (For the proof of the theorem it is not necessary to construct both wuC-series simultanuously but there is no extra effort in doing so and it might be useful elsewhere.)

Lemma 0.2. *Let X be L-embedded, let (x_n) be a sequence in X and let (t_n) be a sequence in X_{s}. Furthermore, suppose that $x + x_{\mathrm{s}}$ is a weak*-cluster point of the x_n and that, along the same filter on $\mathbb{N}$, $u + u_{\mathrm{s}}$ is a weak*-cluster point of the t_n (with $x, u \in X$, $x_{\mathrm{s}}, u_{\mathrm{s}} \in X_{\mathrm{s}}$). Let finally $x^*, u^* \in X^*$ be normalized elements.*

Then there is a sequence (n_k) in $\mathbb{N}$ and there are two wuC-series $\sum x_k^*$ and $\sum u_k^*$ in X^* such that

$$t_{n_k}(x_k^*) = 0 \quad \text{for all } k \in \mathbb{N}, \tag{0.1}$$

$$\lim_k x_k^*(x_{n_k}) = x_{\mathrm{s}}(x^*), \tag{0.2}$$

$$\lim_k t_{n_k}(u_k^*) = u^*(u), \tag{0.3}$$

$$u_k^*(x_{n_k}) = 0 \quad \text{for all } k \in \mathbb{N}. \tag{0.4}$$

Proof. Let $1 > \varepsilon > 0$ and let (ε_j) be a sequence of numbers decreasing to zero such that $0 < \varepsilon_j < 1$ and $\prod_{j=1}^\infty (1 + \varepsilon_j) < 1 + \varepsilon$.

By induction over $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we shall construct four sequences $(x_k^*)_{k\in\mathbb{N}_0}$, $(y_k^*)_{k\in\mathbb{N}_0}$, $(u_k^*)_{k\in\mathbb{N}_0}$ and $(v_k^*)_{k\in\mathbb{N}_0}$ in X^* (of which the first members x_0^*, y_0^* u_0^*, and v_0^* are auxiliary elements used only for the induction) and an increasing sequence (n_k) of indices such that, for all (real or complex) scalars α_j

and with $\beta = x_{\mathrm{s}}(x^*)$, $\gamma = u^*(u)$, the following conditions hold for all $k \in \mathbb{N}_0$:

$$x_0^* = 0, \qquad \|y_0^*\| = 1, \tag{0.5}$$

$$u_0^* = 0, \qquad \|v_0^*\| = 1, \tag{0.6}$$

$$\Big\|\alpha_0 y_k^* + \sum_{j=1}^k \alpha_j x_j^*\Big\| \leq \Big(\prod_{j=1}^k (1+\varepsilon_j)\Big) \max_{0\leq j\leq k} |\alpha_j|, \quad \text{if } k \geq 1, \tag{0.7}$$

$$\Big\|\alpha_0 v_k^* + \sum_{j=1}^k \alpha_j u_j^*\Big\| \leq \Big(\prod_{j=1}^k (1+\varepsilon_j)\Big) \max_{0\leq j\leq k} |\alpha_j|, \quad \text{if } k \geq 1, \tag{0.8}$$

$$t_{n_k}(x_k^*) = 0, \tag{0.9}$$

$$u_k^*(x_{n_k}) = 0, \tag{0.10}$$

$$y_k^*(x) = 0, \quad \text{and} \quad x_{\mathrm{s}}(y_k^*) = \beta, \tag{0.11}$$

$$u_{\mathrm{s}}(v_k^*) = 0, \quad \text{and} \quad v_k^*(u) = \gamma, \tag{0.12}$$

$$|x_k^*(x_{n_k}) - \beta| < \varepsilon_k, \quad \text{if } k \geq 1, \tag{0.13}$$

$$|t_{n_k}(u_k^*) - \gamma| < \varepsilon_k, \quad \text{if } k \geq 1. \tag{0.14}$$

We set $n_0 = 1$, $x_0^* = 0$, $y_0^* = x^*$, $u_0^* = 0$ and $v_0^* = u^*$.
For the following it is useful to recall some properties of P: The restriction of P^* to X^* is an isometric isomorphism from X^* onto $X_{\mathrm{s}}^\perp$ with $(P^*y^*)_{|X} = y^*$ for all $y^* \in X^*$, Q is a contractive projection and $X^{***} = X_{\mathrm{s}}^\perp \oplus_\infty X^\perp$ (where $X^\perp$ is the annihilator of X in X^{***}).

For the induction step suppose now that $x_0^*, \ldots, x_k^*$, $y_0^*, \ldots, y_k^*$, $u_0^*, \ldots, u_k^*$, $v_0^*, \ldots, v_k^*$ and $n_0, \ldots, n_k$ have been constructed and satisfy conditions (0.5) - (0.14). Since $x + x_{\mathrm{s}}$ is a weak*-cluster point of the x_n and $u + u_{\mathrm{s}}$ is a weak*-cluster point of the t_n along the same filter there is an index n_{k+1} such that

$$|x_{\mathrm{s}}(y_k^*) - y_k^*(x_{n_{k+1}} - x)| < \varepsilon_{k+1}, \tag{0.15}$$

$$|t_{n_{k+1}}(v_k^*) - (u + u_{\mathrm{s}})(v_k^*)| < \varepsilon_{k+1}, \tag{0.16}$$

Put

$$\begin{aligned} E &= \mathrm{lin}(\{x^*, x_0^*, \ldots, x_k^*, y_k^*, P^*x_0^*, \ldots, P^*x_k^*, P^*y_k^*, \\ &\qquad u^*, u_0^*, \ldots, u_k^*, v_k^*, P^*u_0^*, \ldots, P^*u_k^*, P^*v_k^*\}) \subset X^{***}, \\ F &= \mathrm{lin}(\{x_{n_{k+1}}, t_{n_{k+1}}, x, x_{\mathrm{s}}, u, u_{\mathrm{s}}\}) \subset X^{**}. \end{aligned}$$

Clearly $Q^*x_j^*$, $Q^*y_k^*, Q^*u_j^*$, $Q^*v_k^* \in E$ for $0 \leq j \leq k$. By the principle of local reflexivity there is an operator $R: E \to X^*$ such that

$$\|Re^{***}\| \leq (1 + \varepsilon_{k+1})\|e^{***}\|, \tag{0.17}$$

$$f^{**}(Re^{***}) = e^{***}(f^{**}), \tag{0.18}$$

$$R_{|E\cap X^*} = \mathrm{id}_{E\cap X^*} \tag{0.19}$$

for all $e^{***} \in E$ and $f^{**} \in F$.

We define

$$x^*_{k+1} = RP^* y^*_k \quad \text{and} \quad y^*_{k+1} = RQ^* y^*_k, \tag{0.20}$$

$$u^*_{k+1} = RQ^* v^*_k \quad \text{and} \quad v^*_{k+1} = RP^* v^*_k. \tag{0.21}$$

In the following we use the convention $\sum_{j=1}^{0}(\cdots) = 0$. Then we have that

$$\begin{aligned}
\alpha_0 y^*_{k+1} + \sum_{j=1}^{k+1} \alpha_j x^*_j &= R\Big(Q^*(\alpha_0 y^*_k + \sum_{j=1}^{k} \alpha_j x^*_j) + P^*(\alpha_{k+1} y^*_k + \sum_{j=1}^{k} \alpha_j x^*_j)\Big),\\
\alpha_0 v^*_{k+1} + \sum_{j=1}^{k+1} \alpha_j u^*_j &= R\Big(P^*(\alpha_0 v^*_k + \sum_{j=1}^{k} \alpha_j u^*_j) + Q^*(\alpha_{k+1} v^*_k + \sum_{j=1}^{k} \alpha_j u^*_j)\Big).
\end{aligned}$$

Now (0.7) (for $k+1$ instead of k) can be seen as follows:

$$\begin{aligned}
&\Big\|\alpha_0 y^*_{k+1} + \sum_{j=1}^{k+1} \alpha_j x^*_j\Big\| \leq \\
&\overset{(0.17)}{\leq} \quad (1+\varepsilon_{k+1})\Big\|Q^*(\alpha_0 y^*_k + \sum_{j=1}^{k} \alpha_j x^*_j) + P^*(\alpha_{k+1} y^*_k + \sum_{j=1}^{k} \alpha_j x^*_j)\Big\|\\
&= \quad (1+\varepsilon_{k+1}) \max\Big\{\Big\|Q^*(\alpha_0 y^*_k + \sum_{j=1}^{k} \alpha_j x^*_j)\Big\|, \Big\|P^*(\alpha_{k+1} y^*_k + \sum_{j=1}^{k} \alpha_j x^*_j)\Big\|\Big\}\\
&\leq \quad (1+\varepsilon_{k+1}) \max\Big\{\Big\|\alpha_0 y^*_k + \sum_{j=1}^{k} \alpha_j x^*_j\Big\|, \Big\|\alpha_{k+1} y^*_k + \sum_{j=1}^{k} \alpha_j x^*_j\Big\|\Big\}\\
&\leq \quad \Big(\prod_{j=1}^{k+1}(1+\varepsilon_j)\Big) \max\{\max_{0\leq j\leq k} |\alpha_j|, \max_{1\leq j\leq k+1} |\alpha_j|\}\\
&= \quad \Big(\prod_{j=1}^{k+1}(1+\varepsilon_j)\Big) \max_{0\leq j\leq k+1} |\alpha_j|
\end{aligned}$$

where the last inequality comes from (0.5) if $k = 0$, and from (0.7), if $k \geq 1$.

Likewise, (0.8) (for $k+1$ instead of k) is proved.

The conditions (0.9) and (0.11) (for $k+1$ instead of k) are easy to verify because $Pt_{n_{k+1}} = 0$, $Qx = 0$ and $Qx_{\mathrm{s}} = x_{\mathrm{s}}$ thus, by (0.18)

$$t_{n_{k+1}}(x^*_{k+1}) = Pt_{n_{k+1}}(y^*_k) = 0,$$

$$y^*_{k+1}(x) = Qx(y^*_k) = 0 \quad \text{and} \quad x_{\mathrm{s}}(y^*_{k+1}) = Q^* y^*_k(x_{\mathrm{s}}) = x_{\mathrm{s}}(y^*_k) = \beta.$$

In a similar way we obtain (0.10) and (0.12) (for $k+1$ instead of k) by $u^*_{k+1}(x_{n_{k+1}}) = RQ^* v^*_k(x_{n_{k+1}}) = Qx_{n_{k+1}}(v^*_k) = 0$, $u_{\mathrm{s}}(v^*_{k+1}) = Pu_{\mathrm{s}}(v^*_k) = 0$ and $v^*_{k+1}(u) = v^*_k(u) = \gamma$.

Finally, we have

$$x^*_{k+1}(x_{n_{k+1}}) - \beta = y^*_k(x_{n_{k+1}}) - \beta = y^*_k(x_{n_{k+1}} - x) - x_{\mathrm{s}}(y^*_k)$$

by (0.11) whence (0.13) for $k+1$ by (0.15). Analogously, we get (0.14) for $k+1$ via (0.16) and $t_{n_{k+1}}(u^*_{k+1}) = t_{n_{k+1}}(v^*_k)$ and $(u+u_{\rm s})(v^*_k) = \gamma$ by (0.12).
This ends the induction and the lemma follows immediately. □

Corollary 0.3. *The complementary space $X_{\rm s}$ of an L-embedded Banach space X is weak*-sequentially closed.*

Proof. Suppose that (s_n) is a sequence in $X_{\rm s}$ that weak*-converges to $v+v_{\rm s}$. Let $u^* \in X^*$ be normalized, set $t_n = s_n - v_{\rm s}$. We apply the lemma to (t_n) with $u = v$, $u_{\rm s} = 0$ and $x_n = u$ and define a sequence (μ_n) of finitely additive measures on the subsets of $\mathbb{N}$ by $\mu_n(A) = (t_n - u)(\sum_{k\in A} u^*_k)$ for all $A \subset \mathbb{N}$ where $\sum_{k\in A} u^*_k \in X^*$ is to be understood in the weak*-topology of X^* and where the u^*_k are given by the lemma. Then $\mu_n(A) \to 0$ for all $A \subset \mathbb{N}$ and by Phillips' original lemma we get

$$|t_{n_k}(u^*_k)| \overset{(0.4)}{=} |(t_{n_k} - u)(u^*_k)| \le \sum_j |(t_{n_k} - u)(u^*_j)| = \sum_j |\mu_{n_k}(\{j\})| \to 0.$$

Thus $u^*(u) = 0$ by (0.3) and $u = 0$ because u^* was arbitrary in the unit sphere of X^*. Hence (t_{n_k}) weak*-converges to 0 which is enough to see that (s_n) weak*-converges to $v_{\rm s}$ in $X_{\rm s}$. □

Proof. Proof of the theorem: Let X be an L-embedded Banach space with L-projection P. Suppose that the sequence (x^{**}_n) is weak*-null and that $x^{**}_n = x_n + t_n$ with $x_n = Px^{**}_n$. Let x^* be a normalized element of X. The sequence (x_n) is bounded and admits a weak*-cluster point $x + x_{\rm s}$. We use the lemma, this time with the wuC-series $\sum x^*_k$, like in the proof of the corollary and define a sequence (μ_n) of finitely additive measures on the subsets of $\mathbb{N}$ by $\mu_n(A) = x^{**}_n(\sum_{k\in A} x^*_k)$ for all $A \subset \mathbb{N}$. Then $\mu_n(A) \to 0$ for all $A \subset \mathbb{N}$ and by (0.1) and Phillips' original lemma we get

$$|x^*_k(x_{n_k})| = |x^{**}_{n_k}(x^*_k)| \le \sum_j |x^{**}_{n_k}(x^*_j)| = \sum_j |\mu_{n_k}(\{j\})| \to 0.$$

Thus $x_{\rm s}(x^*) = 0$ by (0.2) and $x_{\rm s} = 0$ because x^* was arbitrary in the unit sphere of X^*. It follows that each weak*-cluster point of the set consisting of the x_n lies in X. Hence this set is relatively weakly sequentially compact by the theorem of Eberlein-Šmulian. If x is the limit of a weakly convergent sequence (x_{n_m}) then (t_{n_m}) weak*-converges to $-x$. Hence $x = 0$ by the corollary. This shows that the sequence (x_n) is weakly null and proves the theorem. □

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